Solutions

4.4: Game Theory
Payoff Matrices, Strategies, and Expected Payoffs

In life, it often occurs that you are faced with making a decision or choosing a strategy from several possible choices. For instance, whether to invest in stocks or bonds, whether to cut prices of a product you are selling, or what offensive play to choose in a football game. In all of these instances, your success depends on a variable outcome and sometimes even on someone else's strategy. We model situations like these using matrices, matrix algebras and other techniques. The study of situations like these is called game theory.

Payoff Matrix and Expected Payoff. Consider the thrilling game of "Rock, Paper, Scissors." It is so thrilling that you and your opponent have decided to make a betting game out of, in which the loser must pay the winner $1. We call such a game a two-person zero-sum game because there are two players and each player's loss is equal to the other player's gain. We can represent the game by a matrix, called the payoff matrix.

$$\begin{array}{cc} & \mathbf{B} \\ & \begin{array}{ccc} r & p & s \end{array} \\ \mathbf{A}\ \begin{array}{c} r \\ p \\ s \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

Player A's options, of moves, are listed on the left, while player B's moves are listed on the top. You can think of A as playing the rows and B as playing the columns. In each round of the game, the way a player chooses a move is called a strategy. A player using a pure strategy makes the same move every round, while a player using a mixed strategy chooses each move a certain percentage of the time. Our ultimate goal is to determine which strategy is best for each player and to calculate the expected payoff resulting from a pair of strategies.

Example 1. Let P be the payoff matrix for "rock, paper, scissors" given above. The strategies are probability distribution, so they must add to 1

(a) Suppose player A, the row player, plays all three moves equally often. Write player A's strategy as a row vector. (That is, a matrix with only one row.)

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\frac{1}{3}(1,1,1)$

(b) Suppose player B, the column player, plays paper 75% of the time and scissors 25% of the time. Write her strategy as a column vector. (That is, a matrix with only one column.)

$$\begin{array}{cc} \begin{pmatrix} 0 \\ 3/4 \\ 1/4 \end{pmatrix} & \begin{array}{c} r \\ p \\ s \end{array} \end{array}$$

Example 2. Consider the game given by the payoff matrix

$$P = \begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} \end{array}.$$

Player A picks a move at random, choosing to play the first option p 75% of the time and q the other 25%. Player B also picks a move at random, choosing the first option a 20% of the time and b the other 80%. On average, how much does A expect to win or lose?

Expected Payoff = (Row Strategy) $\begin{pmatrix} \text{Payoff} \\ \text{Matrix} \end{pmatrix}$ $\begin{pmatrix} \text{Column} \\ \text{Strategy} \end{pmatrix}$

$$E = rPc$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}$$

Exercise 1. You and your friend have come up with the following simple game to pass the time: at each round, you simultaneously call "heads" or "tails." If you have both called the same thing, your friend wins 1 points; if your calls differ, you win 1 point. Set up the payoff matrix for this game.

$$\begin{array}{ccc} & & \text{Your friend} \\ & & \begin{array}{cc} H & T \end{array} \\ \text{You} & \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

Counter-Strategies

Example 3. Commercial TV station RTV and cultural station CTV are competing for viewers in the Tuesday prime-time 9-10 pm time slot. RTV is trying to decide whether to show a sitcom, a docudrama, a reality show, or a movie, while CTV is thinking about either a nature documentary, a symphony concert, a ballet, or an opera. A television rating company estimates the payoffs for the various alternatives as follows. Each point indicates a shift of 1000 viewers from one channel to the other.

		CTV			
		Nature Doc(x)	Symphony(y)	Ballet(z)	Opera(w)
RTV	Sitcom(x)	2	1	-2	2
	Docudrama(y)	-1	1	-1	2
	Reality Show(z)	-2	0	0	1
	Movie(w)	3	1	-1	1

(a) If RTV notices that CTV is showing nature documentaries half the time and symphonies the other half, what would RTV's best strategy be, and how many viewers would it gain if it followed this strategy?

(b) If, on the other hand, CTV notices that RTV is showing docudramas half the time and reality shows the other half, what would CTV's best strategy be, and how many viewers would it gain if it followed this strategy?

(a) $E = rPc = \begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$

$= (1.5x - z + 2w)$ with the constraint $x+y+z+w=1$.

Maximized when $w=1$. Thus RTV will gain 2000 viewers from CTV if they play only movies.

(b) $E = rPc = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

$= (-1.5x + 0.5y - 0.5z + 1.5w)$ with the constraint $x+y+z+w=1$.

Minimized when $x=1$. Thus CTV will gain 1500 viewers from RTV if they play only nature documentaries.

Exercise 2. Bored with the game from exercise 1, you decide to use the following variation instead: If you both call "heads" your friend wins 2 points; if you both call "tails" your friend wins 1 point; if your calls differ, then you win 2 points if you call "heads," and 1 point if you called "tails." Set up the payoff matrix for this variation of the game.

Your friend

$$\text{You} \quad \begin{array}{c} \\ H \\ T \end{array} \begin{array}{c} \begin{array}{cc} H & T \end{array} \\ \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \end{array}$$

Critical Thinking.

(a) In exercise 2. what would you say is your optimal strategy? and why?

This is for you to answer.

(b) What criterion are you using to say that it is the optimal strategy?

There are many ways to answer this. We will define a specific one on the next worksheet called the Minimax Criterion.

Exercise 3. (Betting) When you bet on a racehorse with odds of $m - n$, you stand to win m dollars for every bet of n dollars if your horse wins; for instance, if the horse you bet is running at 5-2 and wins, you will win \$5 for every \$2 you bet. (Thus a \$2 bet will return \$7.) Here are some odds from a 1992 race at Belmont Park, NY. The favoriate at 5-2 was Pleasant Tap. Thunder Rumble was running at 7-2, while Strike the gold was running at 4-1. Set up the payoff matrix for yourself assuming you are making a \$10 bet on one of these horses. The payoffs are your winnings. (If your horse does not win, you lose your entire bet. Of course, it is possible for none of your horses to win.)

P = Pleasant Tap

T = Thunder Rumble

S = Strike

N = None of the above

Winner

$$\text{You} \quad \begin{array}{c} \\ P \\ T \\ S \\ N \end{array} \begin{array}{c} \begin{array}{cccc} P & T & S & N \end{array} \\ \begin{pmatrix} 25 & -10 & -10 & -10 \\ -10 & 35 & -10 & -10 \\ -10 & -10 & 40 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$